Lagrangian-averaged Large Eddy Simulations for fluid/magnetofluid turbulence

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Outline

Turbulence is nonlinear

Incompressible fluid/magnetofluid equations

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho_0} \nabla P + \mathbf{j} \times \mathbf{b} + \mathcal{F} + \nu \nabla^2 \mathbf{v}$$

$$\partial_t \mathbf{b} + \mathbf{v} \cdot \nabla \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{v} + \eta \nabla^2 \mathbf{b}$$

$$\nabla \cdot \mathbf{v} = 0, \qquad \boldsymbol{\omega} = \nabla \times \mathbf{v}, \qquad \nabla \cdot \mathbf{b} = 0, \qquad \mathbf{j} = \nabla \times \mathbf{b}$$

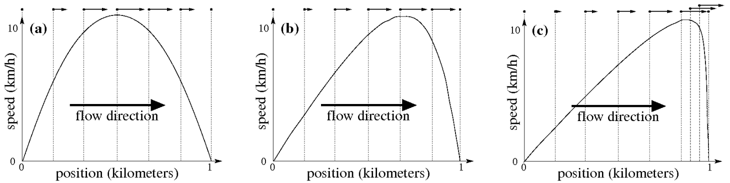
$$\mathbf{b} = \mathbf{B}/\sqrt{\mu_0 \rho_0}, \qquad \partial_t \rho = 0$$

$$\frac{[V]^2[L]^{-1}}{\nu [L]^2 [V]} \sim Re \equiv \frac{v_{r.m.s.} L}{\nu}$$

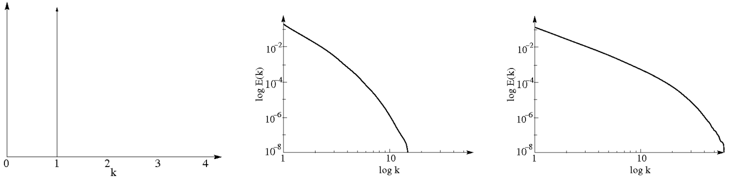
$$Re_M \equiv \frac{v_{r.m.s.} L}{\eta}$$

Turbulence has a long range of scales

Cascade to small scales example: $\partial_t v + v\partial_x v = \nu\partial_{xx} v$



0 minutes 1.25 minutes 2.5 minutes

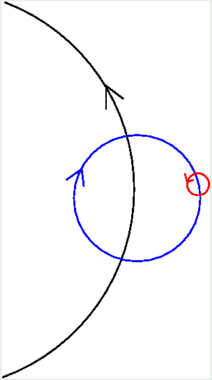


wavenumber wavenumber wavenumber

Kolmogorov picture of the "direct" cascade (K41 theory)

Assumptions

- spectral locality



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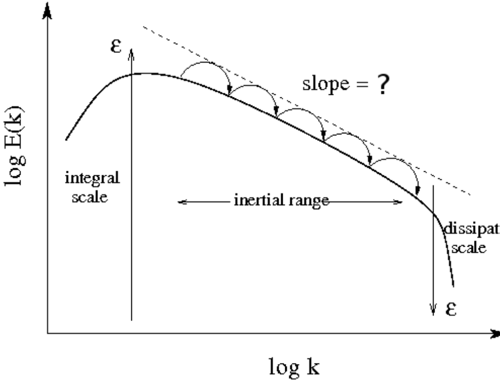
$$\partial_t \hat{\mathbf{v}}(\mathbf{k}) + \mathfrak{F}\left[\mathbf{v}\cdot\nabla\mathbf{v} + \nabla P\right](\mathbf{k}) = 0 + \cancel{\hat{\mathbf{F}}(\mathbf{k}) - \nu|\mathbf{k}|^2\hat{\mathbf{v}}(\mathbf{k})}$$

- no forcing
- no dissipation
- $\Rightarrow$ “inertial” range



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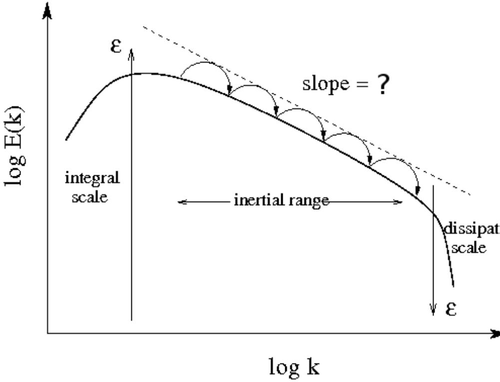
$\varepsilon \sim \partial_t E_K \equiv$
$\partial_t \frac{1}{2} v^2 = -\mathbf{v} \cdot \nabla \left(\frac{1}{2} v^2 + P \right)$



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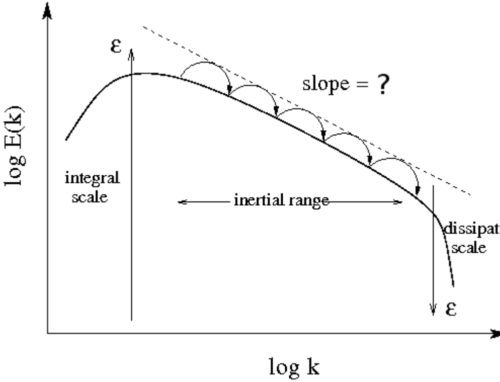
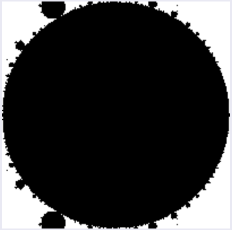
- Constant flux

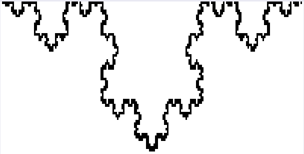
$\varepsilon \sim v^3 \frac{1}{l}$
$v^3 \sim \varepsilon l$



Exact self-similarity

Mandelbrot set & Koch curve





Wikipedia

Scaling relation for self-similar function

$$f(\lambda x) = \lambda^h f(x)$$
$$\rightarrow f(x) = Ax^h$$

Kolmogorov 1941

Assumptions

1. spectral locality
2. self-similarity:

$$\langle \delta v_{\|}(\lambda l) \rangle = \lambda^h \langle \delta v_{\|}(l) \rangle$$
$$v^2 \sim \varepsilon l^{2/3}$$
$$\Rightarrow E_K(k) \propto \varepsilon^{2/3} k^{-5/3}$$

Kolmogorov 1941

Assumptions

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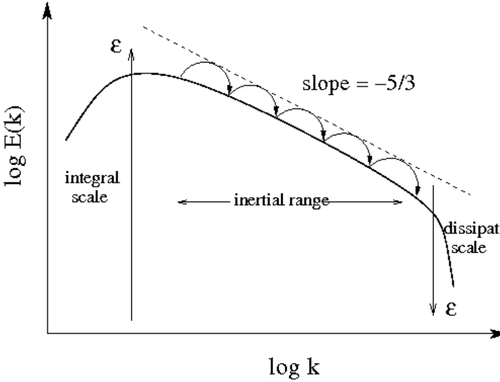

K41: We can't simulate *(much)* turbulence!

How long is the cascade?

- Until all ε is dissipated:

$$\frac{\varepsilon}{\nu} = \int^{k_\nu} k^2 E_K(k) dk \sim \epsilon^{2/3} k_\nu^{4/3}$$

$$l_\nu = \frac{2\pi}{k_\nu} \sim \left(\frac{\varepsilon}{\nu^3}\right)^{-1/4} \sim Re^{-3/4}$$

$$dof \equiv (L/l_\nu)^3 \sim Re^{9/4}$$

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K41: $dof \propto Re^{9/4}$

Supergranule: $Re_M = \frac{vL}{\eta} \sim 3 \cdot 10^6$

$\rightarrow 300,000^3$ simulation

4096^3 Earth Simulator (Kaneda et al 2003)

$\rightarrow$ year 2040 to resolve B-field

$Re \sim 10^{11} \rightarrow$ year 2080 for v

Corona: $Re_M \sim [10^8, 10^{12}]$

(Aschwanden 2006)

Solar wind: $Re_M \sim 10^{11}$

(Weygand et al. 2007)

Interstellar medium: $Re_M \sim 10^{11}$

(Zweibel 1999)

What can we do about it?

Modeling

- Temporal filtering: Reynolds averaging
- Spatial filtering: Large Eddy Simulations (LES)
  - Implicit
    - Moderate *Re* models high *Re*?
    - Dissipative numerical techniques
  - Explicit
    - Devise a model of the un-resolved scales

What can we do about it?

Large Eddy Simulations (LES)

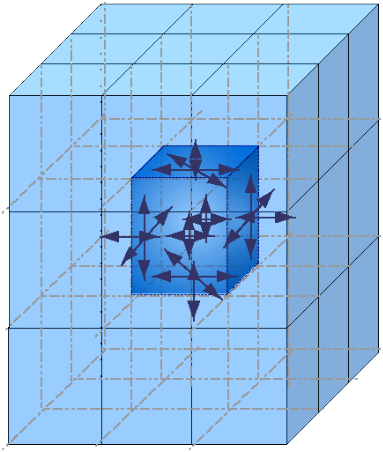
$L : \mathbf{z} \to \bar{\mathbf{z}}$

$$\partial_t \bar{\mathbf{v}} + \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} = -\nabla \bar{P} + \nu \nabla^2 \bar{\mathbf{v}} - \nabla \cdot \underline{\tau}$$

divergence of subgrid stress (SGS) tensor: $\nabla \cdot \underline{\tau} = \nabla \cdot (\overline{\mathbf{v}\mathbf{v}} - \bar{\mathbf{v}}\bar{\mathbf{v}})$

LES in real space

Modeling the effect of unresolved scales

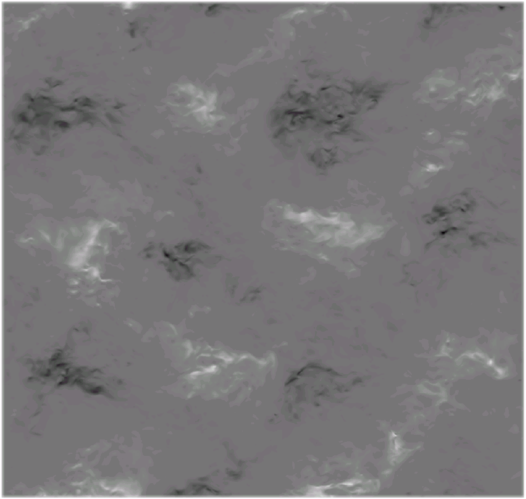


divergence of subgrid stress (SGS) tensor

$$\nabla \cdot \underline{\tau} = \nabla \cdot (\overline{\mathbf{v}\mathbf{v}} - \bar{\mathbf{v}}\bar{\mathbf{v}})$$

Turbulence is Intermittent, *not* self-similar

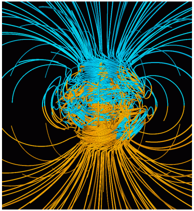
Worry about "back-scatter" from unresolved scales in LES



Direct, local cascade is an incomplete picture

Worry about interactions with un-resolved scales

- Turbulence is *non*-local
  - Nonlocal transfers for fluids ($Re^{-1/2}$) (Alekaxis et al. 2005, 2006)
  - MHD *very* nonlocal (Alfvén waves)
- Self-organization: "inverse cascade"
  - Quasi 2D nonconducting fluid - inverse cascade of energy
  - MHD - inverse cascade of magnetic helicity, $\int \mathbf{a} \cdot \mathbf{b} dV$



www.es.ucsc.edu/ glatz/geodynamo.html

LES: limited success

Series of *ad-hoc* models

- Smagorinsky/eddy-viscosity: $\tau_{ij} = -2(C_S\alpha)^2|\underline{\underline{S}}|S_{ij}$, only dissipative: no back-scatter; inhibits transition to turbulence; excessively dissipative near walls
- Dynamic: $C_S(\mathbf{x}, t)$ by assuming self-similarity at test filter scale; improved results but destabilizes simulations
- Similarity model: $\underline{\underline{\tau}}$ is self-similar back-scatter; inadequate dissipation, inaccurate *a posteriori* results
- Leonard tensor-diffusivity/Clark: generic α^2 term of $\nabla \cdot \underline{\underline{\tau}}$ excellent *a priori*: back-scatter, globally dissipative; *a posteriori* needs extra dissipation to perform

No general LES for MHD

Challenges

- Eddy-viscosity $\leftrightarrow k^{-5/3}$ (Chollet & Lesieur 1981) *not -3/2*
- E_K & E_M *not* conserved quantities
- Spectrally **nonlocal** interactions between large scale of one field and small scale of the other (Alexakis et al. 2005; Alexakis 2007)
- Unresolved **v** & **b** interactions
- Many regimes – no generally applicable MHD-LES

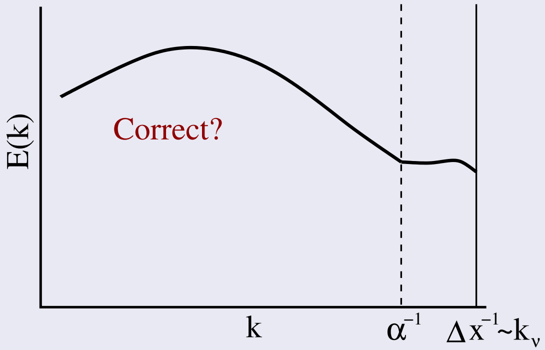
No general LES for MHD

Existing Models

- Dissipative LES (Theobald et al 1994)
  - Ignore sub-filter scale energy exchanges
  - Assumes energy spectra of non-conserved quantities
- Dissipative LES (Zhou et al 2002)
  - non-helical, stationary MHD
  - $k^{-5/3}$ and fixed ratio of energies
- Cross-helicity model (Müller & Carati 2002)
  - Assumes alignment between the fields
  - Reduced intermittency
- Low Re_M LES (Ponty et al 2004)
- Hyper-resistivity (not LES - Haugen & Brandenburg 2006)
  - Requires recalibration of length scales to known DNS

1 - Do the models work?

Do sub-filter-scale physics reproduce super-filter-scale properties?



Lagrangian-averaged Navier-Stokes (LANS, $\alpha-$model)

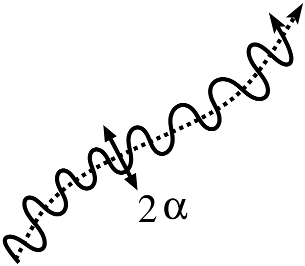
Camassa et al. 1993, Holm et al. 1998, Chen et al. 1998

What is the model?

1. Generalized Lagrangian mean (Andrews & McIntyre 1978)
2. Taylor's frozen-in-turbulence

Mathematically

- Retains Hamiltonian structure
- Preserves Kelvin's theorem, small-scale circulation
- Conservation of energy, helicity (H^1_α *not* L^2: $\frac{1}{2}\langle \bar{\mathbf{v}} \cdot \mathbf{v} \rangle$ *not* $\frac{1}{2}\langle v^2 \rangle$)



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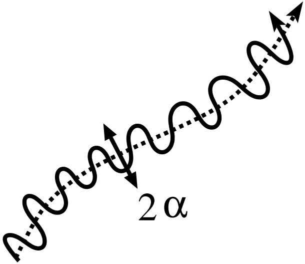
Physically

- Retains non-local large-small interactions
- Limits small local interactions
- Reduces flux of energy in sub$-\alpha$ scales



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Equations

$$\partial_t v_i + \partial_j(\bar{v}_j v_i) + \partial_i \pi + v_j \partial_i \bar{v}_j = \nu \partial_{jj} v_i$$

$$\partial_j v_j = \partial_j \bar{v}_j = 0$$

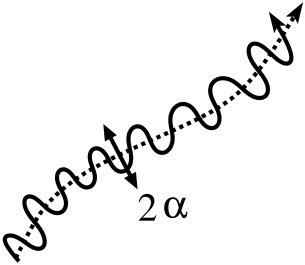
Filter: $v_i = (1 - \alpha^2 \partial_{jj}) \bar{v}_i$

LES form

$$\partial_t \bar{v}_i + \partial_j(\bar{v}_j \bar{v}_i) + \partial_i \bar{P} + \partial_j \bar{\tau}^{\alpha}_{ij} = \nu \partial_{jj} \bar{v}_i$$

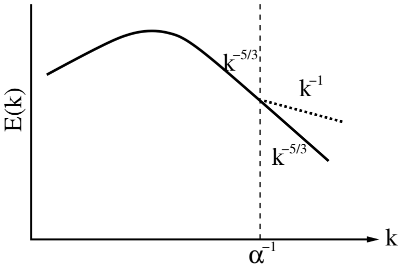
SGS:

$$\bar{\tau}^{\alpha}_{ij} = (1 - \alpha^2 \partial_{jj})^{-1} \alpha^2 (\partial_m \bar{v}_i \partial_m \bar{v}_j + \partial_m \bar{v}_i \partial_j \bar{v}_m - \partial_i \bar{v}_m \partial_j \bar{v}_m)$$



LANS α – *model*: How does it work?

$H^1_\alpha \sim k^{-1}$ (Holm 2002)



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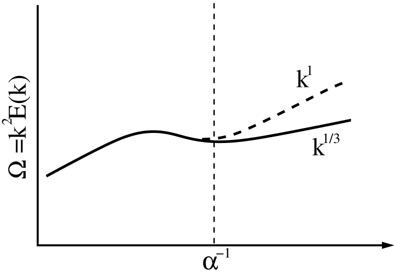

Dissipates faster in k

$-\frac{dE}{dt} = \varepsilon = 2\nu\Omega \sim \frac{1}{Re}\int^{k_\nu} k^2 E(k) dk$

$E(k)dk \sim \varepsilon^\gamma k^\beta$

$k_\nu \sim Re^{1/(3+\beta)} \quad \beta = -5/3$ or -1

$dof_\alpha \sim \alpha^{-1} Re^{3/2}$

(predicted Foias et. al 2001, confirmed Graham et al. 2007)

$dof_{NS} \sim Re^{9/4}$

LANS α – *model*: At what *Re*?

Great at moderate *Re*

- Better than dynamic eddy viscosity ($Re_\lambda \approx 220$, Mohseni et al. 2003)
- Better than dynamic mixed (similarity) eddy viscosity ($Re \approx 50$, Geurts & Holm 2006)

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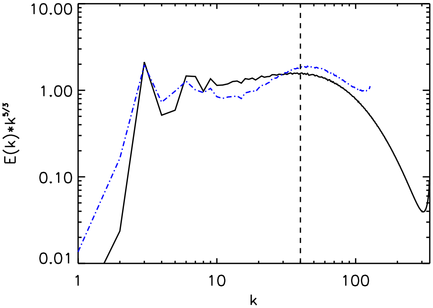
Forced TG $k = 2$, $Re \approx 3300$

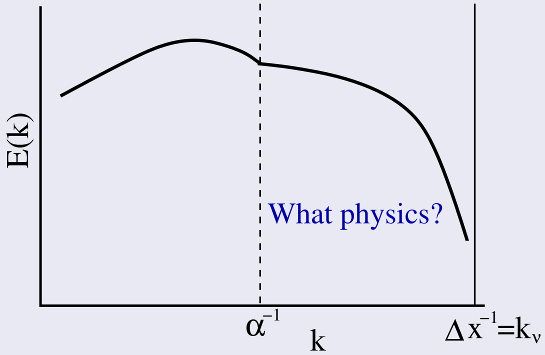


Navier-Stokes 1024^3

LANS 384^3, $\alpha = 2\pi/40$

2 - **HOW** do the models work?

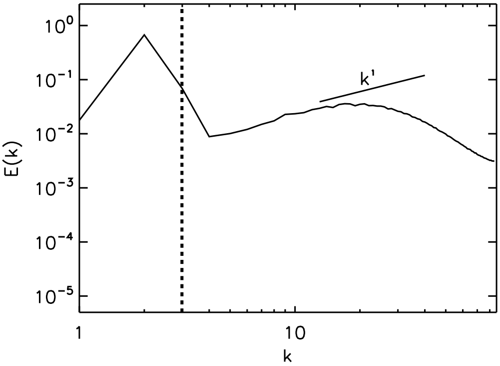
What are the sub-filter-scale physics?



LANS α – *model*: How does it fail?

Graham et al. PRE **76**, 056310 (2007)

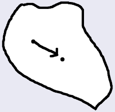
LANS 256^3, $\alpha = 2\pi/3$



Forced TG $k = 2$, $Re \approx 8000$

Rigid bodies

$\delta\bar{\mathbf{v}}(\mathbf{l}) = \mathbf{\Omega} \times \mathbf{l}$



$\delta\bar{v}_{||}(l) = \delta\bar{\mathbf{v}}(\mathbf{l})\cdot\mathbf{l}/l = 0$

$\langle(\delta\bar{v}_{||})^3\rangle = 0$

$\delta\bar{v}^2 \sim l^0$

$\bar{v} \sim \alpha^{-2}k^{-2}v$

$E_\alpha(k)k \sim \bar{v}v \sim k^2$

$E_\alpha(k) \sim k^1$

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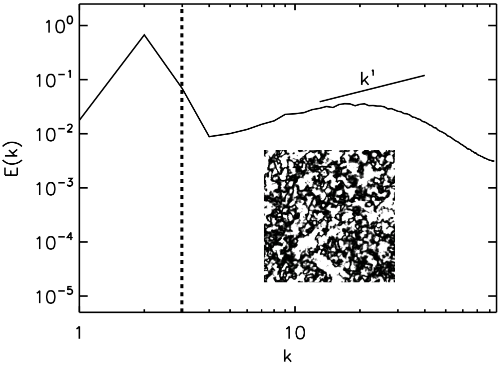

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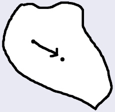

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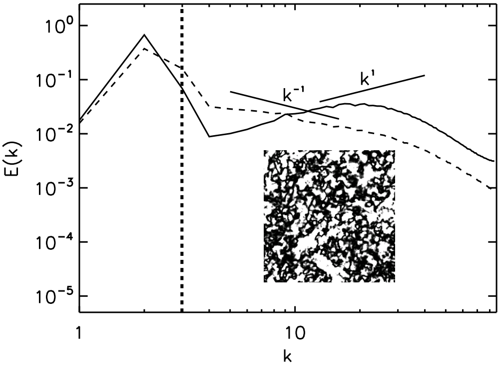

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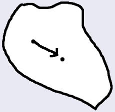

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How to get rid of rigid bodies?

Change regularization

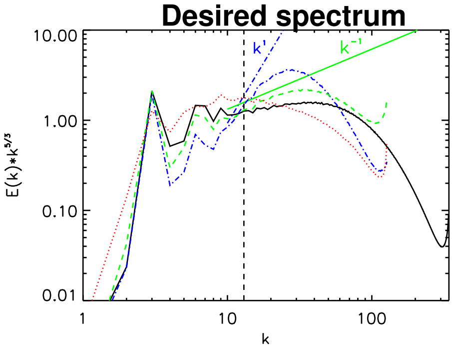
- Truncate LANS$-\alpha$
  $\bar{\tau}^{\alpha}_{ij} = (1-\alpha^2\partial_{jj})^{-1}\alpha^2(\partial_m\bar{v}_i\partial_m\bar{v}_j + \partial_m\bar{v}_i\partial_j\bar{v}_m - \partial_i\bar{v}_m\partial_j\bar{v}_m)$
- 1 term Clark$-\alpha$ (Cao et al. 2005)
- 2 terms Leray$-\alpha$ (Geurts & Holm 2002, 2003, 2006; Cheskidov et al. 2005)
- Conserves H^1_α, L^2 energy but *not* helicity, circulation

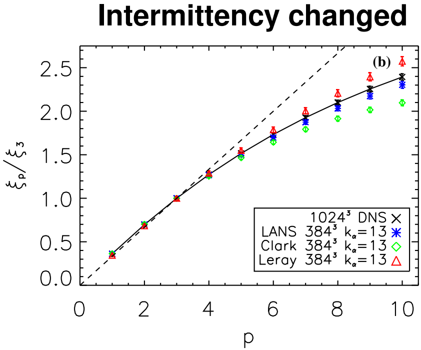
Clark$-\alpha$, Leray$-\alpha$: Sub-filter-scale properties

Graham et al. Phys. Fluids **20**, 035107 (2008)

Navier-Stokes 1024^3

Clark$-\alpha$, Leray$-\alpha$, LANS 384^3, $\alpha = 2\pi/13$





Forced TG $k = 2$, $Re \approx 3300$, $Re_\lambda \approx 790$

What about MHD?

Circumvents rigid body formation?

- Source term in Kelvin's circulation theorem
  $\frac{d}{dt}\Gamma = \frac{d}{dt}\oint_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r} = \oint_{\mathcal{C}} \mathbf{j} \times \mathbf{b} \cdot d\mathbf{r}$
- Spectrally **nonlocal** interactions between large scale of one field and small scale of the other (Alexakis et al. 2005; Alexakis 2007)

LAMHD$-\alpha$ (MHD$-\alpha$)

Holm 2002, Montgomery & Pouquet 2002

Equations

$$\partial_t \mathbf{v} + \boldsymbol{\omega} \times \bar{\mathbf{v}} = \mathbf{j} \times \bar{\mathbf{b}} - \boldsymbol{\nabla}\pi + \nu\nabla^2\mathbf{v}$$
$$\partial_t \bar{\mathbf{b}} = \boldsymbol{\nabla} \times (\bar{\mathbf{v}} \times \bar{\mathbf{b}}) + \eta\nabla^2\mathbf{b}$$
$$\boldsymbol{\nabla} \cdot \mathbf{v} = \boldsymbol{\nabla} \cdot \bar{\mathbf{v}} = \boldsymbol{\nabla} \cdot \mathbf{b} = \boldsymbol{\nabla} \cdot \bar{\mathbf{b}} = 0$$

Filter: $\mathbf{v} = (1 - \alpha^2\nabla^2)\bar{\mathbf{v}}$, $\mathbf{b} = (1 - \alpha^2\nabla^2)\bar{\mathbf{b}}$

Properties

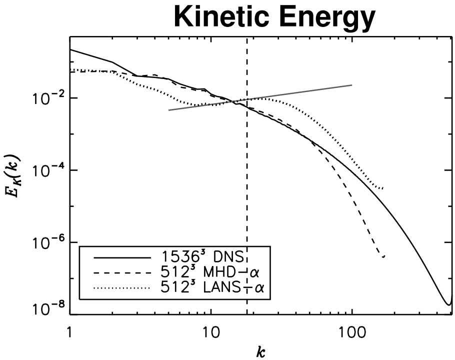
- Math
  - Preserves ideal MHD invariants (H^1_α *not* L^2)
  - Alfvén's theorem
- Physics
  - Supports Alfvén waves at all scales
  - Wavelengths $< \alpha$: slows & damps

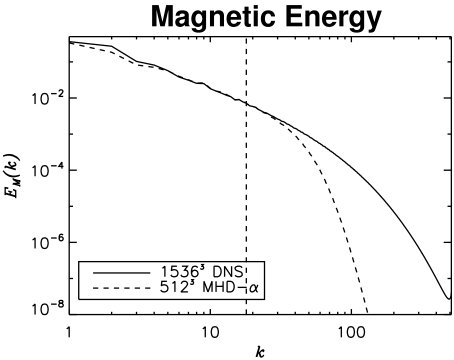
LAMHD$-\alpha$: No positive power laws; No contamination

Graham et al. PRE **80**, 016313 (2009)

MHD 1536^3

LANS, LAMHD 512^3, $\alpha = 2\pi/18$

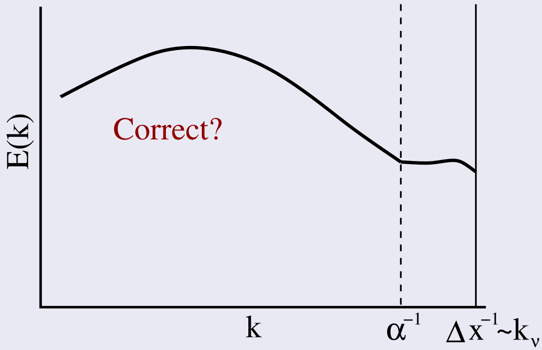




Decay ABC $k \in [1, 4]$ + *noise*, $Re \approx 9200$, $Re_\lambda \approx 1100$

1 - Do the models work?

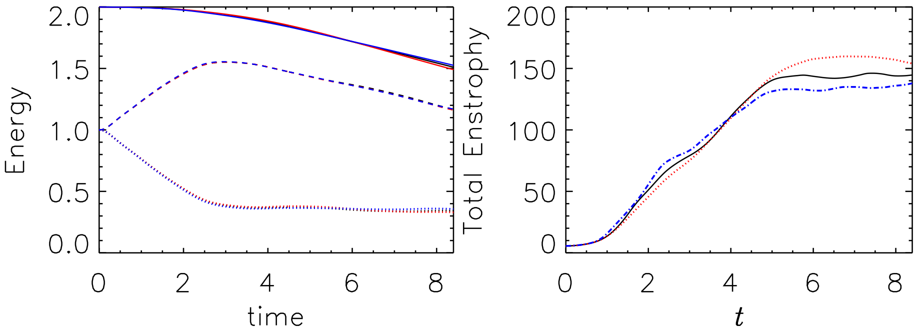
Do sub-filter-scale physics reproduce super-filter-scale properties?



MHD$-\alpha$ SGS test: Global quantities

DNS 1024^3

MHD 168^3, LAMHD 168^3 $\alpha = 2\pi/28$

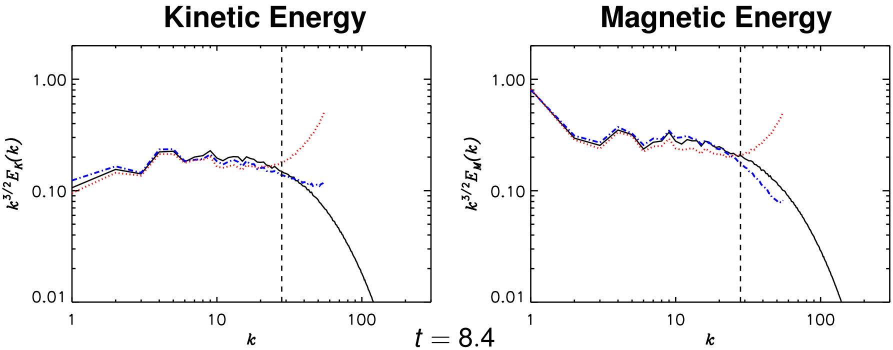


Decay ABC $k \in [1, 4]$, $Re \approx 3300$

MHD$-\alpha$ SGS test: Better spectra

DNS 1024^3

MHD 168^3, LAMHD 168^3 $\alpha = 2\pi/28$

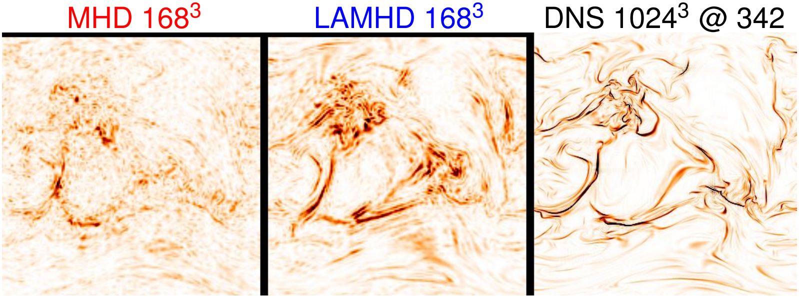


$t = 8.4$

Decay ABC $k \in [1, 4]$, $Re \approx 3300$

MHD$-\alpha$ SGS test: Captures current sheets

Square current, j^2



$t = 8.4$

Conclusions

Lagrangian-averaged Navier-Stokes α

- Conserves small-scale circulation
- Prohibits local small-scale to small-scale interactions
- Develops rigid bodies $\rightarrow$ spectral contamination

Lagrangian-averaged Magnetohydrodynamics α

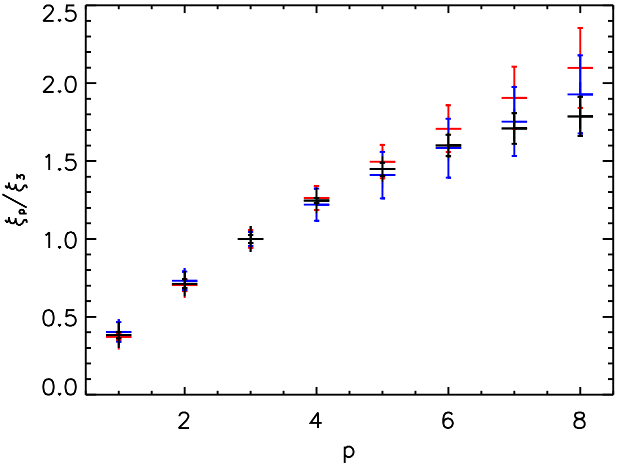
- Lorentz force is source of circulation and conduit for *nonlocal* interactions
- Only damps small-wavelength Alfvén waves & local small-scale interactions
- May be viable SGS

Previous tests

2D†	time evolution of energies	✓
	time evolution of cross-helicity	$\approx$
	energy spectra	+
	dynamic alignment	$\approx$
	PDFs	except tails
	inverse cascade of vector potential	$<$
3D‡	time evolution of energies	✓
	time evolution of magnetic helicity	$\approx$
	energy spectra	✓
	dynamic alignment	$<$
	inverse cascade of magnetic helicity	$<$
	dynamo	✓

† Mininni et al. *Phys. Fluids* **17**, 035112 (2005). ‡ Mininni et al. *Phys. Rev. E* **71**, 046304 (2005), Ponty et al. *Phys. Rev. Lett.* **94**, 164502

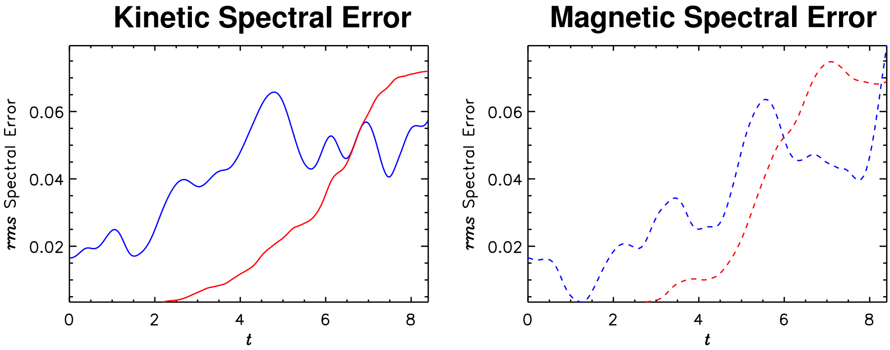
MHD$-\alpha$ SGS test: Better intermittency



MHD$-\alpha$ SGS test: Better spectra

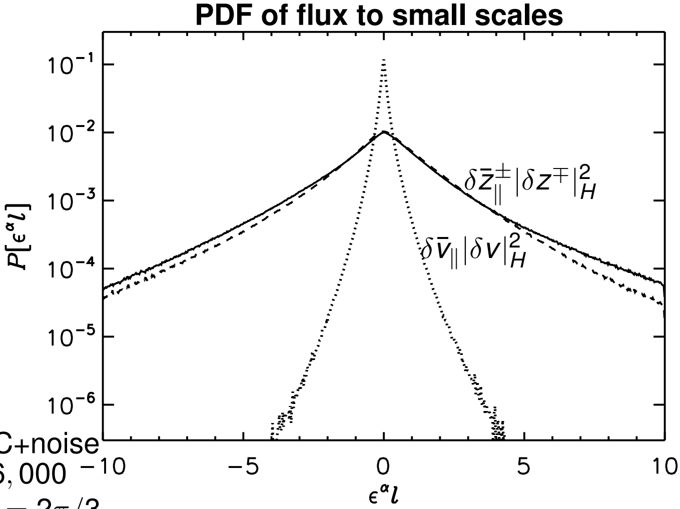
MHD 168^3, LAMHD 168^3 $\alpha = 2\pi/28$

ϵ_0^b, Meyers et al. 2006



LAMHD$-\alpha$: No rigid bodies

Graham et al. PRE **80**, 016313 (2009)



Decay ABC+noise

$Re \approx 26,000$

256^3, $\alpha = 2\pi/3$